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Abstract

We introduce the notion of twisted balanced metrics. These metrics are induced from specific projective embeddings and can be understood as zeros of a certain moment map. We prove that on a polarized manifold, twisted constant scalar curvature metrics are limits of twisted balanced metrics, extending a result of S.K. Donaldson and T. Mabuchi.

Let M be a smooth projective manifold of complex dimension n. Let L be an ample line bundle on M, thus giving a *polarization* of the considered manifold. In that paper, we consider an extra data T, a *twisting*, where T is a line bundle on M. Let h_T be a smooth hermitian metric on T and denote its curvature $\frac{1}{2}\alpha$. Let h_L be a smooth hermitian metric on L whose curvature ω is a Kähler form. We are interested in the following twisted constant scalar curvature equation,

$$Scal(\omega) - \Lambda_\omega \alpha = C_\alpha \tag{1}$$

where C_α is a topological constant equal to $4n\pi \frac{(c_1(M)-2c_1(T))\cdot c_1(L)^{n-1}([M])}{c_1(L)^n([M])}$. A solution to Equation (1) is said to be an α-twisted constant scalar curvature Kähler metric (α*-twisted cscK metric* in short).

This equation was introduced by J. Fine in [Fi1, Fi2] and studied recently by J. Stoppa in order to understand the behavior of K-stability under deformations of polarizations [St1, St2]. We believe that it has others applications, since it appears naturally in various problems of complex geometry as we shall see later.

Let now introduce some notations. Let $Aut(M)$ be the group of holomorphic automorphisms of M. Then, the group of $\widehat{Aut}(M,L)$ of holomorphic automorphisms of (M,L) is formed of couples $(\kappa, \widehat{\kappa})$ where κ is a biholomorphism of M and $\widehat{\kappa}$ is a biholomorphim of the bundle $\pi_L : L \to M$ covering κ, i.e $\pi_L \circ \widehat{\kappa} = \kappa \circ \pi_L$. The kernel of the projection on the first factor $\widehat{Aut}(M,L) \twoheadrightarrow Aut(M)$ is composed of the trivial automorphisms $\mathbb{C}^*$ and we will denote $Aut(M,L) = \widehat{Aut}(M,L)/\mathbb{C}^*$. The following two conditions will appear naturally in the sequel :

(C_1) The Lie algebra $Lie(Aut(M,L))$ is trivial and T is semi-positive, α is a pointwise semi-positive $(1,1)$-form on M.

(C_2) T is ample and α is a positive $(1,1)$-form on M.

Let us give now some explanations about our condition on the Lie algebra $Lie(Aut(M,L))$. An element of $Lie(\widehat{Aut}(M,L))$ can be described as the real part of a $\mathbb{C}^*$-invariant holomorphic vector field of L. Of course, there is a canonical map

$$\tau : Lie(\widehat{Aut}(M,L)) \to Lie(Aut(M))$$

by pushing down via π_L such a vector field seen as an element of $Aut(L)$. Then $Lie(Aut(M,L))$ is trivial if and only if τ has trivial image. This latter condition appeared in the work of Donaldson who identified $Lie(Aut(M,L))$ with the kernel of the Lichnérowicz operator.

Notation. *In all the following, $Met(\Xi)$ will denote the space of smooth hermitian metrics on the bundle or vector space Ξ. Moreover J will be the complex structure on M and $\mathrm{Diff}(M)$ the space of diffeomorphisms of M in a fixed homotopy class. For a smooth hermitian metric h on a line bundle, $c_1(h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(h)$ represents its curvature.*

In a first part, using a technical result about Bergman kernels, we will describe the notion of twisted balanced metrics from a symplectic point of view. Then, we study the convergence of a sequence of twisted balanced metrics when there exists a solution to Equation (1). Our main result is Theorem 2.

1 Twisted balanced metrics

In this section, we introduce a notion of *twisted* balanced metrics adapted to Equation (1). Our goal is to provide natural candidates for being quantizations of the solutions to Equation (1).

First of all, we will need the following technical result about asymptotic of Bergman functions. For k sufficiently large, the line bundle $L^k \otimes T^{-1}$ is very ample. Since M is compact, the vector space $H^0(M, L^k \otimes T^{-1})$ has finite dimension and we denote

$$N_k = \dim H^0(M, L^k \otimes T^{-1}).$$

We can consider the Bergman kernel B over $M \times M$ as the kernel of the L^2 projection π from $\mathrm{C}^\infty(M, L^k \otimes T^{-1})$ to $H^0(M, L^k \otimes T^{-1})$ with respect to the natural L^2 metric induced by $h_L^k \otimes h_T^{-1}$ and the volume form $\frac{\omega^n}{n!}$. Actually, one has for any $f \in \mathrm{C}^\infty(M, L^k \otimes T^{-1})$ and $x \in M$,

$$\pi(f)(x) = \int_M B(x,y) f(y) \frac{\omega^n(y)}{n!}.$$

We can express the restriction of the Bergman kernel over the diagonal, that we shall call the Bergman function. In particular, one can write

$$B(x) = B(x,x) = \sum_{i=1}^{N_k} |s_i|^2_{h_L^k \otimes h_T^{-1}}(x)$$

where the sections $(s_i)_{i=1,..,N_k}$ form an orthonormal basis of $H^0(M, L^k \otimes T^{-1})$ with respect to the L^2 inner product defined previously:

$$\langle .,. \rangle = \int_M h_L^k \otimes h_T^{-1}(.,.) \frac{\omega^n}{n!}.$$

Clearly, the Bergman kernel is independent of the choice of the orthonormal basis. Now, one obtains the asymptotic behavior of $B(x)$ when k tends to infinity.

Theorem 1.1. *With our previous notations, one has for k large enough,*

$$\frac{1}{k^n}\Big\| B(x) - \Big(k^n + \frac{k^{n-1}}{2}\left(Scal(\omega) - \Lambda_\omega \alpha\right)\Big)\Big\|_{\mathrm{C}^r(\omega)} \leq \frac{\gamma}{k^2}$$

where γ is a constant depending on r, h_L, h_T. In particular if h_L varies in a compact subset of $Met(L)$ and has positive curvature, then γ depends only on r and h_T.

Proof. This is essentially a consequence of [Lu, Wa], and we refer to [M-M] as a general survey on this topic. In particular a proof can be found with [M-M, Theorem 4.1.2] but for the sake of clearness, we will sketch the computation of the terms of the asymptotic. The key point is that the problem is purely local. It is clear that

$$B(x) = \sup_{s\in H^0(M,L^k\otimes T^{-1})} \frac{|s(x)|^2_{h_L^k\otimes h_T^{-1}}}{\|s\|^2} \tag{2}$$

and one can reduce the problem to construct the section that represents this supremum at $x \in M$. Let us call this section $s_{extr(x)}$, the extremal section at x, which is unique up to scaling. Now, one can choose a smooth section $s_0 \in \mathrm{C}^\infty(M, L^k \otimes T^{-1})$ such that s_0 is concentrated in L^2 norm on a small geodesic ball $\mathtt{B}$ of radius $\frac{\log(k)}{\sqrt{k}}$ around x. Without loss of generality, one can fix $|s_0(x)|^2_{h_L^k\otimes h_T^{-1}} = 1$. Furthermore, using Hörmander's $\bar\partial$-estimates, one can modify s_0 to make it holomorphic. Hörmander's estimates can be applied because of the positivity of $L^k \otimes T^{-1}$ for large k. This gives $s_{extr(x)} \in H^0(M, L^k \otimes T^{-1})$ with $|s_{extr(x)}(x)|^2_{h_L^k\otimes h_T^{-1}} = 1$. Hence, from (2), we are lead to compute the L^2 norm of s_{extr}. In order to do that, we specify some appropriate coordinates. On one hand, using Böchner coordinates, one can write locally $h_L = e^{-\phi_{L,x}}$ where $\phi_{L,x}$ is plurisubharmonic with

$$\phi_{L,x}(z) = |z|^2 - \frac{1}{4}R_{i\bar{j}k\bar{l}} z_i \bar{z}_j z_k \bar{z}_l + O(|z|^5).$$

Here $R_{i\bar{j}k\bar{l}}$ denotes the Riemannian curvature tensor of the Riemannian metric $g_{i\bar{j}}$ induced by $c_1(h_L)$ on M. On another hand, in the same coordinates and thanks to some affine transformations, $h_T^{-1} = e^{-\psi_{T,x}}$ where the potential $\psi_{T,x}$ satisfies

$$e^{-\psi_{T,x}} = 1 - \sum_{1\leq k,l\leq n} c_1(h_T^{-1})_{k\bar{l}} z_k \bar{z}_l + O(|z|^3).$$

Let's denote $dV_0 = \left(\frac{\sqrt{-1}}{2\pi}\right)^n dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n$. Now, one finds explicitly when k tends to infinity,

$$\begin{aligned}
\|s_{extr(x)}\|^2 &\sim \int_{\mathtt{B}\left(x,\frac{\log(k)}{\sqrt{k}}\right)} |s_{extr(x)}|^2_{h_L^k\otimes h_T^{-1}} \frac{\omega^n}{n!} \\
&\sim \int_{\mathtt{B}\left(x,\frac{\log(k)}{\sqrt{k}}\right)} e^{-k\phi_{L,x}-\psi_{T,x}} \det(g_{i\bar{j}}) \\
&\sim \int_{\mathtt{B}\left(x,\frac{\log(k)}{\sqrt{k}}\right)} e^{-k|z|^2}\left(1+\frac{k}{4}R_{i\bar{j}k\bar{l}}z_i\bar{z}_j z_k\bar{z}_l + O(|z|^5)\right) \\
&\quad \times \left(1-\sum_{1\leq k,l\leq n} c_1(h_T^{-1})_{k\bar{l}} z_k\bar{z}_l + O(|z|^3)\right) e^{-\mathrm{Ric}_{i\bar{j}}z_i\bar{z}_j+O(|z|^3)}dV_0
\end{aligned}$$

Actually the last expression is equal to

$$\int_{\mathtt{B}\left(x,\frac{\log(k)}{\sqrt{k}}\right)} e^{-k|z|^2}\left(1+\frac{k}{4}R_{i\bar{j}k\bar{l}}z_i\bar{z}_j z_k\bar{z}_l \right.$$
$$\left. -\mathrm{Ric}_{i\bar{j}}z_i\bar{z}_j - \sum_{1\leq k,l\leq n} c_1(h_T^{-1})_{k\bar{l}} z_k\bar{z}_l + O(|z|^5)\right) dV_0 + O\left(\frac{1}{k^{n+2}}\right).$$

This can be evaluated using the fact that given f a function on $\{1,..,n\}^p \times \{1,..,n\}^p$,

$$\sum_{I,J}\int_{|z|<\log(k)/\sqrt{k}} f_{I,\bar{J}} z_{i_1}..z_{i_p}\bar{z}_{j_1}..\bar{z}_{j_p}|z|^{2q}e^{-k|z|^2}dV_0 =$$
$$\left(\frac{1}{p!}\sum_I\sum_{\sigma\in\Sigma_p} f_{I,\sigma(\bar{I})}\right)\frac{p!(n+p+q+1)!}{(p+n-1)!k^{n+p+q}} + O(\frac{1}{k^{p'}}),$$

for any $p'>0$. Hence, one gets after removing non symmetric terms (in holomorphic and anti-holomorphic variables)

$$\begin{aligned}
\|s_{extr(x)}\|^2 &= \frac{1}{k^n}+\frac{1}{k^{n+1}}(-Scal(g_{i\bar{j}})) + 2\frac{1}{k^{n+2}}\left(\frac{k}{4}Scal(g_{i\bar{j}})\right) \\
&\quad -\frac{1}{k^{n+1}}\Lambda_\omega\left(-\frac{\alpha}{2}\right) + O\left(\frac{1}{k^{n+2}}\right) \\
&= \frac{1}{k^n} - \frac{1}{2k^{n+1}}\left(Scal(g_{i\bar{j}}) - \Lambda_\omega\alpha\right) + O\left(\frac{1}{k^{n+2}}\right),
\end{aligned}$$

which gives the result. □

We now consider the Bergman function as depending on the choice of the metric h_L. In that context and generalizing the notion of balanced metrics studied by S. Zhang and H. Luo, it is natural to introduce the

Definition 1.1. *A metric h_L is said to be h_T-twisted balanced of order k if the k-th Bergman function associated to it satisfies for all $x\in M$,*

$$B_{h_L,h_T}(x) = \frac{N_k}{Vol(L)}$$

where $Vol(L) = c_1(L)^n([M])$ is the volume of L.

An obvious consequence of Theorem 1.1 is the following result.

Proposition 1.1. *Assume that there exists for all k sufficiently large a metric $h_k \in Met(L^k)$ which is h_T-twisted balanced, and assume that the sequence $(h_k)^{1/k} \in Met(L)$ is convergent in C^∞ topology. Then its limit h_∞ has curvature ω_∞ solution to Equation (1), i.e ω_∞ is an α-twisted cscK metric.*

Remark 1.1. *The reason of our normalization of the form α by a factor $\frac{1}{2}$ is precisely due to the asymptotic expansion of Theorem 1.1 and Equation (1).*

Furthermore, we can see twisted balanced metrics as Fubini-Study metrics, i.e they can be understood as algebraic type metrics. Let us denote the complex vector space

$$V = H^0(M, L^k \otimes T^{-1}).$$

We define the Fubini-Study map

$$FS : Met(V) \to Met(L^k)$$

such that for $H \in Met(V)$, $FS(H)$ is the hermitian metric satisfying for all $x \in M$,

$$\sum_{i=1}^{N_k} |s_i|^2_{FS(H)\otimes h_T^{-1}}(x) = \frac{N_k}{Vol(L)}$$

where $(s_i)_{i=1,..,N_k}$ is an H-orthonormal basis of V. On another hand, one can construct the Hilbertian inner product on V by considering the map

$$Hilb_{h_T} : Met(L^k) \to Met(V)$$

such that

$$Hilb_{h_T}(h) = \int_M h \otimes h_T^{-1}(.,.) \frac{c_1(h^{1/k})^n}{n!}.$$

Then obviously, h_T-twisted balanced maps are fixed points of the map

$$FS \circ Hilb_{h_T} : Met(L^k) \to Met(L^k).$$

This can be rephrased by saying that there exist metrics $H \in Met(V)$ – that we shall call again twisted balanced metrics – satisfying that $FS(H)$ is twisted balanced in the sense of Definition 1.1 or

$$\int_M \langle s_i, s_j \rangle_{FS(H)\otimes h_T^{-1}} \mu_{FS(H)} = \delta_{ij}$$

where $\mu_{FS(H)}$ is the induced Fubini-Study volume form and $(s_i)_{i=1,..,N_k}$ is H-orthonormal. On other words, through the Kodaira embedding

$$\iota : M \hookrightarrow \mathbb{P}(V^*)$$

induced by the sections of $H^0(M, L^k \otimes T^{-1})$, the center of mass of M is trivial.

2 The moment map picture

In this section, following the ideas of Donaldson [Do1], we show that twisted balanced metrics appear as zeros of a certain natural double symplectic quotient.

2.1 The infinite dimensional picture

Given hermitian metrics h_L, h_T on the polarization and the twisting as before, the space $\mathrm{C}^\infty(M, L^k \otimes T^{-1})$ has a natural symplectic form

$$\Omega(\alpha,\beta) = \mathrm{Re}\left(\int_M \langle J\alpha, \beta\rangle_{h_L^k\otimes h_T^{-1}} \frac{\omega^n}{n!}\right)$$

and thus it is natural to consider the moment map associated to the group $\mathcal{G}_k$ of hermitian bundle maps from $L^k \otimes T^{-1}$ to $L^k \otimes T^{-1}$ that preserve the Chern connection induced by h_L and h_T.
Via the classical hamiltonian construction, its Lie algebra can be identified with $\mathrm{C}^\infty_0(M,\mathbb{R})$ the space of smooth functions on M with vanishing integral. Note that $\mathcal{G}_k$ acts as automorphims of $L^k \otimes T^{-1}$ covering the action of elements of $Symp(M,\omega)$, the group of hamiltonian symplectomorphisms preserving the Kähler form ω.
The moment map associated to this action and the symplectic form Ω is described in [Do1, Section 2.1]. This is explicitly given by $\mu : \mathrm{C}^\infty(M, L^k\otimes T^{-1}) \to Lie(\mathcal{G}_k)^*$, where

$$\mu(s) = \frac{-1}{2n} J\nabla_{L^k\otimes T}(s) \wedge \nabla_{L^{-k}\otimes T^{-1}}(s^*) \wedge \frac{\omega^{n-1}}{(n-1)!} + k|s|^2_{h_L^k\otimes h_T^{-1}} \frac{\omega^n}{n!}.$$

Of course, if s is holomorphic with respect to the fixed holomorphic structure on $L^k \otimes T^{-1}$, then the former expression simplifies as

$$\mu(s) = \left(\frac{1}{2}\Delta|s|^2_{h_L^k\otimes h_T^{-1}} + k|s|^2_{h_L^k\otimes h_T^{-1}} - \widehat{s}\right)\frac{\omega^n}{n!}.$$

Here Δ is the Laplace operators acting on functions and one has fixed the constant $\widehat{s}$ to be $\widehat{s} = \frac{1}{Vol(L)}\int_M \left(\frac{1}{2}\Delta|s|^2_{h_L^k\otimes h_T^{-1}} + k|s|^2_{h_L^k\otimes h_T^{-1}}\right)\frac{\omega^n}{n!}$.

On another hand, when acting by $\mathcal{G}_k$, one needs to move the complex structure in order to preserve the holomorphicity property of a section. Thus, it is natural to consider the induced action of $\mathcal{G}_k$ over the space $\mathcal{J}_{int}$ of all ω-compatible complex structure over M (i.e the set of all almost-complex structures such that its Nijenhuis tensor is zero). One can see $\mathcal{J}_{int}$ as the space of sections of a $Sp(2n)/U(n)$-bundle over M. With the complex structure of $Sp(2n,\mathbb{R})/U(n)$ and its natural metric, one obtains using the volume form ω^n, a Kähler structure over the infinite dimensional manifold $\mathcal{J}_{int}$. Note that the group $Symp(M)$ preserves this Kähler structure. It acts on the structure J by

$$\psi(J) = \psi_* J^{-1} \psi_*^{-1}.$$

In particular, it is now easy to check that the space

$$\Upsilon = \{(s_1, ..., s_{N_k}, J) \in \mathrm{C}^\infty(M, L^k\otimes T^{-1})^{N_k} \times \mathcal{J}_{int},\ \text{s.t. } \bar\partial_J s_i = 0, \forall 1 \le i \le N_k\}$$

is preserved by the diagonal action of $\mathcal{G}_k$.

Let us denote $\pi : \Upsilon \to \mathrm{C}^\infty(M, L^k\otimes T^{-1})^{N_k}$ the equivariant projection. Then similarly to what is happening in [Do1, Lemma 12'], π_1 is injective and one can pull-back Ω to the space Υ. The moment map associated to the action of $\mathcal{G}_k$ over Υ is now given by

$$\mu_{\mathcal{G}_k}(s_1, ..., s_{N_k}, J) = \left(\left(\frac{1}{2}\Delta + k\right)\left(\sum_{i=1}^{N_k} |s_i|^2_{h_L^k\otimes h_T^{-1}}\right) - \widehat{s}_k\right)\frac{\omega^n}{n!}.$$

with $\widehat{s}_k = \frac{1}{Vol(L)}\int_M \left(\frac{1}{2}\Delta + k\right)\left(\sum_{i=1}^{N_k}|s_i|^2_{h_L^k\otimes h_T^{-1}}\right)\frac{\omega^n}{n!}$. Moreover, we notice that

$$\mu_{\mathcal{G}_k}(s_1,...,s_{N_k},J) = 0 \tag{3}$$

is equivalent to the condition

$$\sum_{i=1}^{N_k}|s_i|^2_{h_L^k\otimes h_T^{-1}} = \frac{\widehat{s}_k}{k}. \tag{4}$$

This comes by taking the L^2 inner product with eigenfunctions of the Laplacian in (3).

2.2 The double symplectic quotient

We remark now that there is a another natural action on Υ. The special unitary group $SU(N_k)$ is acting over Υ and the associated moment map is just

$$\mu_{SU}(s_1,...,s_{N_k},J) = \frac{\sqrt{-1}}{2}\left(\int_M \langle s_i, s_j\rangle_{h_L^k\otimes h_T^{-1}}\frac{\omega^n}{n!} - \frac{1}{N_k}\sum_{i=1}^{N_k}\|s_i\|^2_{L^2(\omega)}\delta_{ij}\right),$$

whose image lies in the space of trace free matrices. Hence, finding a zero of the moment map μ_{SU} corresponds formally to choosing a basis of orthonormal sections with respect to the inner product induced by h_L, h_T.

The moment map for the action of the product $\mathcal{G}_k \times U(N_k)$ is given by the sum $\mu_{\mathcal{G}_k}\oplus\mu_{SU}$ and of course we can consider the double symplectic quotient

$$\Upsilon//(\mathcal{G}_k\times SU(N_k)) = \frac{\mu_{\mathcal{G}_k}^{-1}(0)\cup\mu_{SU}^{-1}(0)}{\mathcal{G}_k\times SU(N_k)}. \tag{5}$$

This quotient inherits from Marsden-Weinstein theorem a canonical symplectic structure. Given a metric $h \in Met(L^k)$, a zero of the moment map $\mu_{\mathcal{G}_k}\oplus\mu_{SU}$ corresponds to a point $(s_1,...,s_{N_k},J)$ such that the $(s_i)_{i=1,..,N_k}$ form an orthonormal basis of holomorphic sections with respect to $Hilb_{h_T}(h)$ and such that that the function $\sum_{i=1}^{N_k}|s_i|^2_{h\otimes h_T^{-1}}\in \mathrm{C}^\infty(M,\mathbb{R})$ is constant. This is precisely to say that the metric h is h_T-twisted balanced of order k.

Of course, our construction is parallel to the one described to [St1, Section 2]. In that case, if one fixes a complex structure J, compatible with ω, it can be considered the space

$$\widehat{\Upsilon} = \{(f, f^*(J)) \text{ s.t. } f\in \mathrm{Diff}(M) \text{ and } f^*(J) \text{ is } \omega\text{−compatible}\}\subset \mathrm{Diff(M)}\times\mathcal{J}_{int}.$$

Then, by choosing the right symplectic form (depending on α) on $\widehat{\Upsilon}$, one can see that the action of $Symp(M,\omega)$ induces the moment map $\widehat{\mu}:\widehat{\Upsilon}\to \mathrm{C}_0^\infty(M)$, where

$$\widehat{\mu}(f,f^*(J)) = Scal(\omega, f^*(J)) - \Lambda_\omega f^*(\alpha) - c$$

where one has fixed the constant $c = \frac{1}{Vol(L)}\int_M Scal(\omega,f^*(J)) - \Lambda_\omega f^*(\alpha)\frac{\omega^n}{n!}$. With Theorem 1.1 in hand, one can consider our previous construction as a quantization of the one described by Stoppa. We shall see in the following section that this quantization holds at the metric level as expected.

3 Approximation of twisted cscK metrics

In this section we use the double symplectic quotient constructed before to show the convergence of the twisted balanced metrics when there exists a twisted cscK metric *a priori*.

3.1 Gradient flow for finding zeros of the moment map

We briefly present some general results about moment maps. Let G be a compact Lie group acting on a Kähler manifold N and $\nu : N \to Lie(G)^*$ a moment map for the action of G. Assume that G has discrete stabilizers for all points of N. At the point $p \in N$, the infinitesimal action $\sigma_p : \zeta \to \sigma_\zeta(p)$ of G induces an injective map $\theta_p : Lie(G) \to T_pN$ and the operator

$$Q_p = \theta_p^* \theta_p : Lie(G) \to Lie(G)$$

is invertible. Here the adjoint is computed by considering an (invariant) metric on $Lie(G)$ and the metric on N. One can define the operator norm over $Lie(G)$,

$$\Lambda_p^{Lie(G)} = |||Q_p^{-1}|||_{Lie(G)},$$

i.e $\Lambda_p^{Lie(G)}$ is the largest eigenvalue of Q_p^{-1}. This quantity controls the convergence of the gradient flow of the norm square of the momentum map,

$$\frac{\partial \nu(p_t)}{\partial t} = -\nu(p_t).$$

It also gives the distance of the initial point to the zero of the moment map.

Proposition 3.1. *Let $p_0 \in N$. Assume that there exist positive constants r_0, r_1 such that,*

$$|\nu(p_0)| < \frac{r_1}{r_0}, \qquad \Lambda_{e^{i\zeta}p_0}^{Lie(G)} \leq r_0 \quad \forall |\zeta| \leq r_1,$$

then there exists $\eta \in Lie(G)$ such that $|\eta| \leq r_1$ and

$$\nu(e^{i\eta}p_0) = 0,$$

i.e $e^{i\eta}p_0$ is a zero of the moment map ν.

In order to find a zero of the moment map $\mu_{\mathcal{G}_k} \oplus \mu_{SU}$, we proceed in two steps. First, we look for the first symplectic quotient

$$\Upsilon // \mathcal{G}_k.$$

This corresponds to finding a (non necessarily orthonormal) basis $(s_i)_{i=1,..,N_k} \in V$ and a metric h_k such that

$$\sum_{i=1}^{N_k} |s_i|^2_{h_k \otimes h_T^{-1}} = C_k$$

where C_k is a constant depending only on k. Such a metric will be called an *approximate twisted balanced metric.* For the second step, thanks to Proposition 3.1, one deforms an approximate twisted balanced metric using the gradient flow of $|\mu_{SU}|^2$ to obtain a zero of the moment map μ_{SU}, and thus an orthonormal basis of holomorphic sections.

3.2 Construction of a formal solution

In that section, we show how one can build an approximate twisted balanced metric $\widetilde{h}_k \in Met(L^k)$ when one assumes the existence of a twisted cscK metric, that we shall denote $\omega_\infty \in c_1(L)$. We use a deformation type argument.

Let us write $\omega_\infty = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(h_\infty)$. Now, we are seeking to modify h_∞ in order to force the twisted Bergman function to be as close to a constant as we want. We write

$$\widetilde{h}_k = h_\infty\left(1 + \frac{\varpi_1}{k} + \frac{\varpi_2}{k^2} + \frac{\varpi_3}{k^3} + ...\right)$$

and apply Lu-Catlin-Wang asymptotic expansion (Theorem 1.1). Then, at $x \in M$, for any integer $r \geq 1$ and k large enough, our Bergman function satisfies

$$\frac{1}{k^n}B_{\widetilde{h}_k,h_T}(x) = \sum_{i=0}^{r}\frac{a_i(\omega_\infty)}{k^i} + \sum_{i,l=1}^{r}\frac{\widetilde{a}_{i,l}}{k^{i+l}} + O\left(\frac{1}{k^{r+1}}\right) \tag{6}$$

$$= a_0 + \frac{a_1}{k} + \frac{a_2 + \widetilde{a}_{1,1}}{k^2} + \frac{a_3 + \widetilde{a}_{2,1} + \widetilde{a}_{1,2}}{k^3} + ... \tag{7}$$

where the coefficients a_i are polynomial of the curvature tensor of h_∞ and its covariant derivatives and the $\widetilde{a}_{i,l}$ are certain multilinear expressions in the ϖ_l and their covariant derivatives. Moreover, from Theorem 1.1, one has $a_0 = 1$ and

$$a_1 = \frac{1}{2}\left(Scal(\omega_\infty) - \Lambda_{\omega_\infty}\alpha\right) = \frac{C_\alpha}{2} \tag{8}$$

are both constants. Writing

$$N_k = \chi(M, L^k \otimes T^{-1}) = k^n\chi_0 + k^{n-1}\chi_1 + k^{n-2}\chi_2 + ...$$

we see that we are lead to find ϖ_1 such that

$$\widetilde{a}_{1,1}(\varpi_1) = \chi_2 - a_2$$

and more generally for $r > 1$,

$$\widetilde{a}_{1,r}(\varpi_r) = \chi_{r+1} - a_{r+1} - \sum_{l=1}^{r-1}\widetilde{a}_{r+1-l,l}. \tag{9}$$

One key point here is that the terms $\widetilde{a}_{r+1-l,l}$ depend only on $\varpi_1, ..., \varpi_{r-1}$. Moreover, because of (8), each term $\widetilde{a}_{1,r}$ is obtained as the differential $\mathcal{L}_\omega$ of the map

$$\omega \mapsto \frac{1}{2}\left(Scal(\omega) - \Lambda_\omega\alpha\right).$$

Consequently, starting with ϖ_1, one can find the ϖ_r using the implicit function theorem recursively if the RHS of (9) does not lie in the kernel of the operator $\mathcal{L}_\omega$ for any $r \geq 1$. If one considers a small deformation $\omega + i\partial\bar{\partial}\phi = \omega_\phi$, then

$$\int_M \phi \cdot \mathcal{L}_\omega(\phi)\omega_\phi^n = \|\bar{\partial}\nabla^{1,0}\phi\|^2_{L^2(\omega_\phi)} + \langle\partial\phi\wedge\bar{\partial}\phi, \alpha\rangle_{L^2(\omega_\phi)},$$

which shows that the kernel of $\mathcal{L}_\omega$ is trivial if either $Lie(Aut(M,L))$ is trivial (Cf. [Bi, Lemme 1.1]) and α is semi-positive, or if α is positive. We have proved the following

Theorem 1. *Assume that condition* (C_1) *or* (C_2) *holds. Assume the existence of a twisted cscK metric*

$$\omega_\infty = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(h_\infty) \in c_1(L)$$

solution to Equation (1). Then for any $q > 0$ *and* k *sufficiently large, there exist smooth functions* $\varpi_1, ..., \varpi_q$ *and a constant* $c_{q,k}$ *such that the metric*

$$\widetilde{h}_k(.,.) = \left(1 + \sum_{i=1}^{q} \frac{\varpi_i}{k^i}\right) h_\infty(.,.)$$

satisfies for all $x \in M$*,*

$$\frac{1}{k^n} B_{\widetilde{h}_k, h_T}(x) = c_{q,k} + R_r(x)$$

where $R_r = O\left(\frac{1}{k^{r+1}}\right)$.

If one divides $\widetilde{h}_k$ by the positive function $\frac{c_{q,k}+R_r(x)}{N_k}$ (for k large enough), one gets the

Corollary 3.1. *Assume that condition* (C_1) *or* (C_2) *holds, and the existence of a twisted cscK metric. Then, there exists an approximate twisted balanced metric, i.e a zero of the moment map* $\mu_{\mathcal{G}_k}$.

3.3 Construction of a twisted balanced point

We are now using the formalism described in section 3.1 to obtain a twisted balanced metric from an approximate twisted balanced one.

We explain how the estimates of [Do1, Section 3.1],[P-S2, Section 5] can be adapted to our problem.

Estimates for the linearised problem

In order to get some uniform estimates, we shall fix for each k sufficiently large, the metric

$$\widetilde{\omega}_\infty = k\omega_\infty.$$

We shall now define a class of metrics by saying that a metric $\widetilde{\omega} \in c_1(L)$ has R-bounded geometry if

$$\widetilde{\omega} > \frac{1}{R}\widetilde{\omega}_\infty, \qquad \|\widetilde{\omega} - \widetilde{\omega}_\infty\|_{C^4(\widetilde{\omega}_\infty)} < R.$$

Moreover, we will say that a basis $(s_i)_{i=1,..,N_k}$ of V has R-bounded geometry if the curvature of the induced Fubini-Study metric has R-bounded geometry. Firstly, with no substantial modification of the proof of [P-S2, Theorem 2], one obtains the

Proposition 3.2. *Assume condition* (C_1) *holds. Then for any* $R > 1$*, there exist two constants* $C > 0$ *and* $\epsilon < 1/10$ *such that if* $\mathfrak{S} = (s_i)_{i=1,..,N_k}$ *is a basis of* V *with* R*-bounded geometry and* $|||\mu_{SU}(\mathfrak{S})||| < \epsilon$*, one has*

$$\Lambda_{\mathfrak{S}}^{Lie(SU(V))} < Ck^2.$$

Here $|||.|||$ *stands for the operator norm on* $Lie(SU(V))$.

Let us now assume that condition (C_1) is not satisfied. Let $\mathcal{H}$ be the maximal connected algebraic subgroup of $Aut^0(M)$, the connected identity component of the group of holomorphic automorphisms of M. Let $\mathcal{Z}$ be the maximal (algebraic) torus in the center of $\mathcal{H}$ and denote its Lie algebra by $Lie(\mathcal{Z})$. We set $K_k = SU(V)$ and K'_k to be the identity component of the subgroup of stabilizers of K_k at the point $\iota(M)$. We shall consider the following condition introduced in [Ma2].

(C_3) The Lie algebras $Lie(\mathcal{Z})$ and $Lie(K'_k)$ can be identified.

At that stage, we remark that if $Lie(Aut(X,L))$ is not trivial, then the approximate twisted balanced metrics obtained in Corollary 3.1 are all $\mathcal{Z}$-invariant, since $\mathcal{Z}$ is also the identity component of the group of isometries of (M, ω_∞). Using the map $Hilb_{h_T}$, the approximate twisted balanced metrics induce a K'_k-invariant inner product on $Lie(K_k)$. Hence, the vector space $Lie(K_k)$ can be decomposed as

$$Lie(K_k) = Lie(K'_k) \oplus Lie(K'_k)^\perp.$$

The following estimates are essentially contained in [Ma2].

Proposition 3.3. *Assume conditions* (C_2) *and* (C_3) *hold for an integer* k *large enough. Then for any* $R > 1$, *there exist two constants* $C > 0$ *and* $\epsilon < 1/10$ *such that if* $\mathfrak{S} = (s_i)_{i=1,..,N_k}$ *is a basis of* V *with* R*-bounded geometry and* $|||\mu_{SU}(\mathfrak{S})||| < \epsilon$, *one has*

$$\Lambda_{\mathfrak{S}}^{Lie(K'_k)^\perp} < Ck^2.$$

Proof. Firstly, let us consider the sequence of holomorphic vector bundles

$$0 \to TM \to \iota^* T\mathbb{P}(V^*)_{|M} \to TM^\perp \to 0 \tag{10}$$

where $TM^\perp$ is the orthogonal complement of TM in $T\mathbb{P}(V^*)_{|M}$, which can be seen as the normal bundle of M in $\mathbb{P}(V^*)$. From the orthogonal decomposition $T\mathbb{P}(V^*)_{|M} = TM \oplus TM^\perp$, one can write for a vector field X on $\mathbb{P}(V^*)$,

$$X_{|M} = X_{|TM} \oplus X_{|TM^\perp}.$$

Then, for any $\zeta \in Lie(K_k)$, the infinitesimal action σ_ζ induces a vector field X_ζ on $\mathbb{P}(V^*)$ such that its restriction to M will be denoted $X_{\zeta,|M}$. Now, from [Bi, Lemme 2.3] the inequality to prove is just equivalent to

$$|\zeta|^2 \leq Ck^2 \int_M |X_{\zeta,|TM^\perp}|^2_{h_{FS}} \widetilde{\mu}_{FS}$$

for all $\zeta \in Lie(K'_k)^\perp$. Here $\widetilde{\mu}_{FS}$ is the volume form of the Kähler metric $c_1(h_{FS}) \in kc_1(L)$ induced by $\mathfrak{S}$ using the Fubini-Study map. Obviously, this inequality can be deduced from the following three inequalities:

$$|\zeta|^2 \leq \gamma_1 k \|X_{\zeta,|M}\|^2_{L^2(\widetilde{\mu}_{FS})} \tag{11}$$

$$\|X_{\zeta,|M}\|^2_{L^2(\widetilde{\mu}_{FS})} \leq \|X_{\zeta,|TM}\|^2_{L^2(\widetilde{\mu}_{FS})} + \|X_{\zeta,|TM^\perp}\|^2_{L^2(\widetilde{\mu}_{FS})} \tag{12}$$

$$\|X_{\zeta,|TM}\|^2_{L^2(\widetilde{\mu}_{FS})} \leq \gamma_2 k \|X_{\zeta,|TM^\perp}\|^2_{L^2(\widetilde{\mu}_{FS})} \tag{13}$$

where γ_1, γ_2 are constants independent of k. The arguments of [P-S2, Theorem 2] can be applied with no change in order to get (11) and (12). Moreover, from the exact sequence (10), one can derive the following estimate (see [P-S2, (5.16)])

$$\|X_{\zeta,|TM^\perp}\|^2_{L^2(\widetilde{\mu}_{FS})} \geq \gamma_3 \|\bar{\partial} X_{\zeta,|TM^\perp}\|^2_{L^2(\widetilde{\mu}_{FS})}. \tag{14}$$

For $\bar{\partial}$ seen as acting on smooth $(0,1)$-form on M with values in TM, one considers the operator $\Box_{h_{FS}^{1/k}} = \bar{\partial}^{*_{h_{FS}^{1/k}}} \bar{\partial}$. As it is pointed out in [Ma2], the first eigenvalue λ_1 of $\Box_{h_{FS}^{1/k}}$ is bounded from below independently of k since $\mathfrak{S}$ has R-bounded geometry. Moreover, since $\zeta \in \mathit{Lie}(K'_k)^\perp$, $X_{\zeta,TM}$ is orthogonal to the projection on TM of any holomorphic vector field on $\iota(M)$ by condition (C_3). Thus, one has

$$\int_M |\bar{\partial} X_{\zeta,TM}|^2_{h_{FS}^{1/k}} \frac{c_1(h_{FS}^{1/k})^n}{n!} \geq \lambda_1 \int_M |X_{\zeta,TM}|^2_{h_{FS}^{1/k}} \frac{c_1(h_{FS}^{1/k})^n}{n!}.$$

But now, $\bar{\partial}(X_{\zeta,TM} + X_{\zeta,TM^\perp}) = 0$ and thus

$$\|\bar{\partial} X_{\zeta,TM^\perp}\|^2_{L^2(\widetilde{\mu}_{FS})} \geq \frac{\lambda_1}{k} \|X_{\zeta,TM}\|^2_{L^2(\widetilde{\mu}_{FS})}. \tag{15}$$

Finally both (15) and (14) imply Inequality (13). □

This is our main result.

Theorem 2. *Assume that either condition* (C_1) *holds or both conditions* (C_2) *and* (C_3) *hold for a sequence of strictly increasing integers $k_j > k_0$. Assume that there exists a twisted cscK metric solution to Equation (1) in the class $c_1(L)$. Then,*

- *For k_0 large enough, there exists a h_T-twisted balanced metric $\omega_{k_j} \in k_j c_1(L)$,*
- *The sequence $\frac{1}{k_j}\omega_{k_j}$ is convergent when $j \to +\infty$ towards the twisted cscK metric in C^∞-topology.*

Proof. Under condition (C_1), the proof of the Theorem is a consequence of Corollary 3.1, Proposition 3.2 and Proposition 3.1 together with the double symplectic quotient picture. The convergence is C^r topology (for any r) is obtained by the fact that one can choose, up to any order, an approximate twisted balanced metric in Theorem 1 (see [Do1, Proof of Theorem 3]). Finally, the uniqueness of the twisted cscK metric is a consequence of [St1], and one could recover this result in the projective setting by studying the uniqueness of twisted balanced metric up to $SU(V)$ action. The convergence of the twisted balanced metric is clear by construction.
Let us now assume conditions (C_2) and (C_3). The main difference with previous case is that one has to check that the gradient flow of $|\mu_{SU}|^2$ is still converging with the estimate obtained from Proposition 3.3. From [Ma2, Lemma 3.4] and [Ma3, Theorem 3.2], it is sufficient to obtain a zero of the moment map by considering the one parameter subgroups perpendicular to the subgroup of stabilizers (one could also invoke [Si, Proposition 9]). Now, Proposition 3.3 shows the convergence of the gradient flow when one restricts the moment map to $\mathit{Lie}(K'_k)^\perp$ and by condition (C_3), we can conclude. □

4 Further directions

Let us discuss some examples of twisted cscK metrics in the literature (see also [Fi1, Fi2]).

- Let $M \to \mathbb{CP}^1$ be an elliptically fibred K3 surface with 24 singular fibres of type I_1. Then, there is a Weil-Petersson metric ω_{WP} induced from the fibres on $\mathbb{CP}^1$. In [S-T1], it is proved that the Kähler-Ricci flow converges to the McLean's metric satisfying the twisted csck equation

$$\mathrm{Ric}(\omega) = \omega_{WP}.$$

- Let us consider an almost Kähler-Einstein Fano manifold M [Ba]. By definition, it carries for any $0 \leq t < 1$ a Kähler metric $\omega_t \in c_1(M)$ such that

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega_0$$

  If condition (C_3) holds, one can modify our arguments to construct a convergent sequence of twisted balanced metrics for any $0 \leq t < 1$. We don't know if in general such a manifold is balanced in the sense of Zhang-Luo.

- Let us consider M an algebraic manifold with semi-ample canonical line bundle. From the minimal model program, we know that M admits an algebraic fibration $\tau : M \to M_{can}$ over its canonical model M_{can}. We assume that $0 < \dim M_{can} < \dim M$, M_{can} is non singular and the fibre $\tau^{-1}(p)$ is non singular for any $p \in M_{can}$. Thus, each fibre $\tau^{-1}(p)$ is a smooth Calabi-Yau manifold. The L^2 metric on the moduli space of Calabi-Yau manifolds induces a semi-positive Weil-Petersson $(1,1)$-form ω_{WP} on M_{can}. Then, the main result of [S-T2] proves the convergence of the Kähler-Ricci flow in that context and identifies its limit. In that case, it satisfies the following twisted cscK equation on M_{can},

$$\mathrm{Ric}(\omega) = -\omega + \omega_{WP}. \tag{16}$$

  In a forthcoming paper, we shall study the dynamical system $Hilb_{h_T} \circ FS$ in order to construct numerical approximations of the solution to Equation (16) for a minimal elliptic surface, by finding twisted balanced metrics.

Finally, one could consider a slightly more general framework. Assume that $\mathcal{T} = (T_j)$ is a finite family of twistings such that $\frac{1}{2}\alpha_{T_j}$ are the curvature of the hermitian metrics $h_{T_j} \in Met(T_j)$. Then, one can consider the $\mathcal{T}$-twisted balanced metrics ω solution to

$$Scal(\omega) - \sum_j \Lambda_\omega \alpha_{T_j} = C$$

where C is a constant. Then, in view of a generalization of Theorems 1 and 2, conditions (C_1) and (C_2) can be replaced respectively by

(C_1') The Lie algebra $Lie(Aut(M,L))$ is trivial and the T_j are semi-positive i.e for all j, α_{T_j} is a pointwise semi-positive $(1,1)$-form on M.

(C$'_2$) There exists j_0 such that T_{j_0} is a ample and $\alpha_{T_{j_0}}$ is a positive $(1,1)$-form on M. For all $j \neq j_0$, T_j is semi-positive, α_{T_j} is pointwisely semi-positive.

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